APPENDIX B

Allocating Tows to Maximize Profits

1. Introduction

This appendix addresses the issue of modeling a water transportation carrier allocating its tows amongst competing uses in order to maximize its profits. The water carrier is assumed to be a perfect competitor in all input markets facing fixed prices for the quantities of inputs it decides to employ, however, this assumption may be relaxed and the important results still hold. The analysis can readily be extended to the case where the carrier is forced to allocate multiple classes of fixed inputs such as tow boats, hopper barges, covered hopper barges, tank barges, and deck barges amongst competing uses. Once again, the important results still apply.

2. Definitions

Let $x_{j,i}$ (j=1,...,m) (i=1,…,n) represent the quantity of variable input j used in producing output i. These inputs are available to the water carrier in any quantity desired. These inputs are meant to represent those inputs that the carrier can vary during the period of analysis such as labor and fuel. Each output i represents a origin, destination, and commodity combination served by the carrier.

Let y represent the total number of tows available to the water carrier. Here a tow is a stylized unit of barges and a towboat. Let y_i (i=1,…,n) represent the number of tows allocated to producing output i and y represent the total number of tows available to the carrier. Here y is fixed during the period of analysis. Note that $y_1 + y_2 + \ldots + y_n$ # y. The total number of tows allocated to alternative origin, destination, commodity combinations must be less than the number of tows operated by the water carrier.

Let $q_i = f_i(x_{1,i}, ..., x_{m,i}, y_i)$ (i=1,…,n) represent the production function for output i. This formulation models the quantity of output produced in each alternative market as a function of the inputs employed in that market by the water carrier. We assume that all inputs are productive and that the law of diminishing marginal returns eventually applies to all inputs.

Let w_j (j=1, ...,m) be the price of variable input j. These prices are perceived as given to the water carrier and do not vary with the quantities of inputs employed. Then total variable costs may be written as

$$V = x_{1,i}w_1 + x_{2,i}w_2 + ...+ x_{m,i}w_m + \ldots + x_{1,n}w_1 + x_{2,n}w_2 + ...+x_{m,n}w_m.$$

Let F denote the total fixed costs of the water carrier. These costs do not vary with the levels of outputs provided and are meant to represent the costs to the carrier of its fixed inputs.

Finally, let S represent the variable costs of allocating the fixed number of tows between the outputs, that is, $S = S(y_1, \ldots, y_n)$. This function represents the possible costs associated with a reallocation of the water carrier's tows to alternative uses.

Let p_i denote the price of output i (i=1, …,n). These prices are not assumed fixed and can vary with the level of outputs. Denote the inverse demand functions for each output as $p_i = p_i(q_i)$ (i=1, …,n). Further, assume that these inverse demand functions are well behaved with the change in price a non-positive function of the change in quantity demanded.

We may now define the water carrier's profit as $\Pi = p_1q_1 + p_2q_2 + \ldots + p_nq_n - V - S - F$. The carrier's profit maximization problem may be defined as maximize Π subject to $y_1 + y_2 + \ldots + y_n$ # y. In words, the water carrier's objective is to maximize profits by allocating its tows to alternative markets subject to the constraint that the sum of all tows allocated must be less than or equal to the number of tows available to the carrier.

3. Solution and Some Important Results

The technique employed to find the solution of the carrier's profit maximization problem involves the application of the Kuhn-Tucker Theorem for constrained maximization. The Kuhn-Tucker Theorem demonstrates that certain conditions must hold for a function to have a maximal value subject to a set of constraints. The solution technique is summarized below. First, form the Lagrangian function. Second, derive the first order conditions for maximization of the Lagrangian. Third, solve the first order conditions. Finally, interpret the economic content of the first order conditions.

Ignoring the non-negativity constraints associated with the individual variables, the Lagrangian of the profit maximization may be written as

(1) $L = \Pi + \lambda(y - y_1 - y_2 - \ldots - y_n)$.

Here λ is a Lagrange multiplier and represents the shadow price of the constraint that there are only y tows available for the water carrier to allocate in producing the outputs. Define $\varepsilon_i = (Mq_i/Mp_i)(p_iq_i)$ and $\theta_i = [1+(1/\varepsilon_i)]$, (i=1,…,n). ε_i and θ_i denote the own-price elasticty of demand and the change in total revenue per change in unit of output, respectively, for each market.

The Kuhn-Tucker Theorem indicates that the first order conditions for a maximization of this Lagrangian and its associated primal objective function are:

(2a) $p_i\theta_i(Mf_i/Mx_{j,i}) - w_j$ # 0, (j=1, ...,m), (i=1,…,n);

(2b) $[p_i\theta_i(Mf_i/Mx_{j,i}) - w_j]x_{j,i} = 0$, (j=1,...,m), (i=1,…,n);

(3a) $p_i\theta_i(Mf_i/My_i)$ - MS/My_i - λ # 0, (i=1,…,n);

(3b) $[p_i\theta_i(Mf_i/My_i)$ - MS/My_i - $\lambda]y_i$ = 0, (i=1,…,n);

(4a) $y - y_1 - y_2 - \ldots - y_n \ \exists\ 0$, and $(y - y_1 - y_2 - \ldots - y_n)\lambda = 0$; and

(4b) $(y - y_1 - y_2 - \ldots - y_n)\lambda = 0$.

Equations (2a) and (2b) are familiar conditions for profit maximization. They ensure that if a variable input is used in production of an output, it will be used in a quantity such that its marginal revenue product is just equal to the price of the input. Equations (3a) and (3b) are of special interest. Note first, the Lagrange multiplier λ represents the shadow price of tows. It represents the amount that total profits would increase per unit increase in the number of tows available to the carrier. In other words, it is the opportunity cost of tows to the water carrier. If tows are used in producing outputs a and b, that is y_a, y_b>0, then equations (3a) and (3b) imply

(5) $p_a\theta_a(Mf_a/My_a) - MS/My_a = p_b\theta_b(Mf_b/My_b) - MS/My_b = \lambda$.

Equation (5) demonstrates that when scarce tows are used to produce different outputs, the imputed marginal revenue products less the incremental variable allocation costs must be equal across all outputs produced in positive quantities, which in turn is equal to the opportunity cost of tows to the water carrier. If the variable costs of allocating tows between markets are equal, then carriers will allocate tows to alternative markets until the imputed marginal revenue foregone is equal across the alternative outputs. Further yet, if the variable costs of allocating tows between markets is negligible as it likely is for tows, then the imputed marginal revenue between all outputs produced with tows are equal to the opportunity cost of tows. This result demonstrates that water carriers will respond to market prices and elasticities in allocating their tows amongst competing uses. It further demonstrates that the profit maximizing water carrier will allocate it tows in a rational manner, producing profitable outputs and not producing unprofitable outputs.

Note that there is information contained in equation (5) regarding when the water carrier will desire to purchase more tows. If λ is greater than the price of a tow, the carrier can increase profits by adding tows. Consequently, when the opportunity cost of tows (profits foregone) to the water carrier is greater than the price of tows, the water carrier will desire to purchase more tows. When the opportunity cost of tows for the water carrier is less than the price of tows, the water carrier will not desire to add additional tows. Hence, if the price of tows is given to the carrier and the carrier faces decreasing marginal productivity of incremental tows and an elastic demand curve in output markets, then there is a point where no further additions to the fleet of tows will purchased by the carrier. Consequently, system congestion cannot increase without bound. In other words, the economic rationality of shippers and carriers will work together to prevent grid-lock of an existing navigation system.